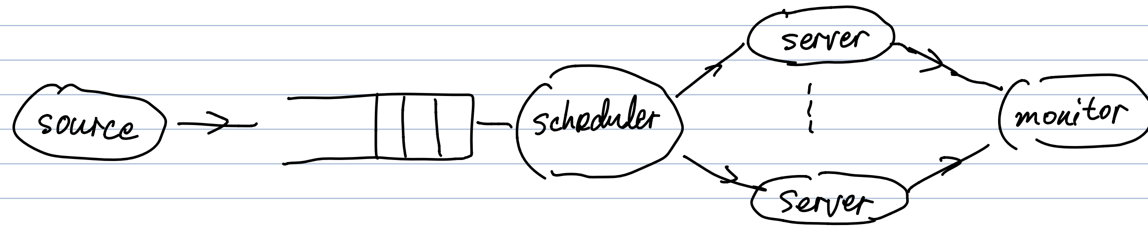
Lecture 15.

Age-optimal Scheduling in Queues. (2)



Theorem:

If service times are i.i.d. across servers & time, then, for all No. of servers $M \geq 1$, all buffer size $B \geq 0$, all generation/arrival times $I = \{S_i, C_i\}$, all policies $\pi \in \Pi$,

$$[\{\Delta_{prmp-LGFS}(t), t \geq 0\} | I] \leq_{st} [\{\Delta_{\pi}(t), t \geq 0\} | I],$$

or for all non-decreasing functionals f

$$E[f(\{\Delta_{prmp-LGFS}(t), t \geq 0\}) | I] \leq E[f(\{\Delta_{\pi}(t), t \geq 0\}) | I],$$

provided expectations exist.

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preemptive LGFS is age-optimal.

Corollary 1: Under condition of the thm,

$$\{\Delta_{prmp-LGFS}(t), t \geq 0\} \leq_{st} \{\Delta_{\pi}(t), t \geq 0\}.$$

$$E[f(\{\Delta_{prmp-LGFS}(t), t \geq 0\})] \leq E[f(\{\Delta_{\pi}(t), t \geq 0\})].$$

Corollary 2.

For in-order arrivals,

preemptive LCFS is age-optimal.

Sample-path arguments of the Thm:

State of the System:

Def: At time $t \geq S_i$, the age of packet i is

$$t - S_i$$

Def: Let $\alpha_{l,\pi}(t)$ be the age of the packet that is being processed by server l. at time t, under policy π.

- If server l is idle at time t,

  $$\alpha_{l,\pi}(t) = \infty.$$

- If a virtual packet with age ∞ is delivered, $\Delta_{\pi}(t)$ is unchanged.

Hence, we can consider that servers are always busy, whereas sometimes the packets being processed have infinite age.

- servers are idle at time 0^-.

$$\alpha_{l,\pi}(0^-) = \infty \quad \forall l, \forall \pi.$$

Def: $\alpha_{[l],\pi}(t)$ is the l-th largest element of the vector $(\alpha_{1,\pi}(t), \cdots, \alpha_{M,\pi}(t))$.

Def: $\vec{V}_\pi(t) = (\Delta_\pi(t), \alpha_{1,\pi}(t), \cdots, \alpha_{M,\pi}(t))$ as system state of policy π.

$\therefore$ $\vec{V}_\pi(0^-)$ is invariant of policy $\pi \in \Pi$.

We use policy P to represent the preemptive LGFS policy.

Compare policy P with any work-conserving policy π, $\pi \in \Pi$.

Lemma 1:

If $\vec{V}_P(0^-) = \vec{V}_\pi(0^-)$ for any work-conserving policy $\pi \in \Pi$.

then for all I.

$$[\{\Delta_P(t), \min\{\Delta_P(t), \alpha_{[1],P}(t)\}, \cdots, \min\{\Delta_P(t), \alpha_{[M],P}(t)\}, t \geq 0\} \mid I]$$

$$\leq_{st} [\{\Delta_\pi(t), \min\{\Delta_\pi(t), \alpha_{[1],\pi}(t)\}, \cdots, \min\{\Delta_\pi(t), \alpha_{[M],\pi}(t)\}, t \geq 0\} \mid I].$$

where $I = \{S_i, C_i\}$.

- intuition:

If a packet with age $\alpha_{[l],\pi}(t)$ is delivered, then the age drops from $\Delta_\pi(t)$, to $\min\{\Delta_\pi(t), \alpha_{[l],\pi}(t)\}$.

Hence, we want $\{\alpha_{[1],\pi}(t), \cdots, \alpha_{[M],\pi}(t)\}$ as small as possible.

Lemma2 (coupling lemma).

For any given I, consider policy P and any work-conserving policy $\pi \in \Pi$. If the service times are i.i.d., then there exist policies P_1 and π_1 in the same probability space, which statisfy the same scheduling discipline with policies P and π, such that:

i $[\{\vec{V}_{P_1}(t), t \geq 0\} \mid I] =_{st} [\{\vec{V}_P(t), t \geq 0\} \mid I]$

ii $[\{\vec{V}_{\pi_1}(t), t \geq 0\} \mid I] =_{st} [\{\vec{V}_\pi(t), t \geq 0\} \mid I]$.

iii If a packet with age $\alpha[l], P_1(t)$ is delivered at time t in policies P_1, then almost surely, a packet with age $\alpha[l], \pi_1(t)$ is delivered at time t in policies π_1, and the reverse is also true.

Proof idea:

① service times are memoryless

② Servers are always busy. $\boxtimes$

Compare policy P_1 and policy π_1.

Lemma 3.

If a packet arrives at time t in both policies P_1 and π_1,

state before arrival   state after arrival

$$\vec{V}_{P_1} = (\Delta_{P_1}, \alpha_{1,P_1}, \cdots, \alpha_{M,P_1}) \Rightarrow \vec{V}_{P_1}' = (\Delta_{P_1}', \alpha_{1,P_1}', \cdots, \alpha_{M,P_1}')$$

$$\vec{V}_{\pi_1} = (\Delta_{\pi_1}, \alpha_{1,\pi_1}, \cdots, \alpha_{M,\pi_1}) \Rightarrow \vec{V}_{\pi_1}' = (\Delta_{\pi_1}', \alpha_{1,\pi_1}', \cdots, \alpha_{M,\pi_1}')$$

If $\quad \Delta_{P_1} \le \Delta_{\pi_1}$

$$\min\{\Delta_{P_1}, \alpha[l], P_1\} \le \min\{\Delta_{\pi_1}, \alpha[l], \pi_1\} \quad \forall l.$$

then $\quad \Delta_{P_1}' \le \Delta_{\pi_1}'$

$$\min\{\Delta_{P_1}', \alpha'[l], P\} \le \min\{\Delta_{\pi}', \alpha'[l], \pi\} \quad \forall l.$$

Lemma 4.

If a packet is delivered at time t in both policies P_1 and π_1,

state before delivery → state after delivery.

$\vec{V}_{P_1} = (\Delta_{P_1}, \alpha_{1,P_1}, \cdots, \alpha_{M,P_1}) \Rightarrow \vec{V}_{P_1}' = (\Delta_{P_1}', \alpha_{1,P_1}', \cdots, \alpha_{M,P_1}')$

$\vec{V}_{\pi_1} = (\Delta_{\pi_1}, \alpha_{1,\pi_1}, \cdots, \alpha_{M,\pi_1}) \Rightarrow \vec{V}_{\pi_1}' = (\Delta_{\pi_1}', \alpha_{1,\pi_1}', \cdots, \alpha_{M,\pi_1}')$

If $\Delta_{P_1} \le \Delta_{\pi_1}$

$\min\{\Delta_{P_1}, \alpha_{[l],P_1}\} \le \min\{\Delta_{\pi_1}, \alpha_{[l],\pi_1}\} \quad \forall l.$

then $\Delta_{P_1}' \le \Delta_{\pi_1}'$

$\min\{\Delta_{P_1}', \alpha_{[l],P}'\} \le \min\{\Delta_{\pi}', \alpha_{[l],\pi}'\} \quad \forall l.$

Reading: Section 2.6.1

Sun, Kadota, Talak, Modiano book 2019.

Proof of Thm:

On a sample path of P_1 & π_1, three case:

1. arrival
2. departure
3. age ↗ with slope $= 1$.

by induction, with probability 1.

$$\Delta_{P_1}(t) \le \Delta_{\pi_1}(t). \qquad \forall t$$

$$\min\{\Delta_{P_1}(t), \alpha_{[l], P_1}(t)\} \le \min\{\Delta_{\pi_1}(t), \alpha_{[l], \pi_1}(t)\} \quad \forall l. \quad \forall t,$$

Using the coupling lemma (Lemma 2).

Lemma 1 is proven.

For non-work-conserving policies, server idling only postpone delivery time, AoI $\nearrow$.

Proof is done $\boxed{\checkmark}$

Reading: Section 2.6.1

Sun, Kadota, Talak, Modiano book 2019.